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Large-N limit of the non-local 2D Yang–Mills and generalized Yang–Mills theories on a cylinder

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Abstract. The large-group behavior of the non-local YM_2's and gYM_2's on a cylinder or a disk is investigated. It is shown that this behavior is similar to that of the corresponding local theory, but with the area of the cylinder replaced by an effective area depending on the dominant representation. The critical areas for non-local YM_2's on a cylinder with some special boundary conditions are also obtained.

1 Introduction

Pure two-dimensional Yang–Mills theories (YM_2) have certain properties, such as invariance under area-preserving diffeomorphisms and lack of any propagating degrees of freedom. There are, however, ways to generalize these theories without losing these properties. One way is by the so-called generalized Yang–Mills theories (gYM_2's). In a YM_2, one starts from a B–F theory in which a Lagrangian of the form $\mathrm{itr}(BF) + \mathrm{tr}(B^2)$ is used. Here F is the field-strength corresponding to the gauge field, and B is an auxiliary field in the adjoint representation of the gauge group. Carrying out a path integral over this field leaves an effective Lagrangian for the gauge field of the form $\mathrm{tr}(F^2)$ [1]. In a gYM_2, on the other hand, one uses an arbitrary class function of the auxiliary field B, instead of $\mathrm{tr}(B^2)$ [2]. In [3] the partition function and the expectation values of the Wilson loops for the gYM_2's were calculated. It is worthy of mention that for gYM_2's, one cannot eliminate the auxiliary field and obtain a Lagrangian for the gauge field. One can, however, use a standard path integration and calculate the observables of the theory. This was done in [4].

The study of the behavior of these theories for large groups is also of interest. This was done in [5,6] for ordinary YM_2 theories and then in [7] for YM_2 and in [8, 9] for gYM_2 theories. It was shown that YM_2's and some classes of gYM_2's have a third-order phase transition in a certain critical area.

In [10] another generalization of YM_2's was introduced, namely to use a non-local action for the auxiliary field. There, the classical behavior, the quantum behavior and the large-group behavior of the system on a sphere were studied.

The large-group behavior of the model on a cylinder or a disk was investigated in [11] for YM_2 and in [12] for gYM_2. Here we want to study the large-group behavior of a non-local YM_2 (or gYM_2) on a cylinder.

The scheme of the present paper is the following. In Sect. 2, it is shown that the dominant representation for large-group models on a cylinder is obtained from a generalized Hopf equation, the same Hopf equation as used for the corresponding local theory. The only difference is that the area of the cylinder is replaced by an effective area involving the dominant representation itself.

In Sect. 3, the critical behavior of the model is investigated, and for some special boundary conditions an equation governing the critical area corresponding to a non-local Yang–Mills theory is obtained.

2 The dominant representation for a large-N non-local generalized Yang–Mills theory on a cylinder

In [10], a non-local Yang–Mills theory was defined through

$$\mathrm{e}^S := \int \mathrm{D}B \exp\left\{\int \mathrm{d}\mu \mathrm{itr}(BF) + w\left[\int \mathrm{d}\mu \Lambda(B)\right]\right\}, \quad (1)$$

where F is the field strength, B is an auxiliary field in the adjoint representation of the gauge group, and Λ is a similarity-invariant function. It was further shown that the wave function for this theory on a cylinder is

$$Z(U_1, U_2) = \sum_R \chi_R(U_1^{-1})\chi_R(U_2^{-1}) \exp\{w[C_\Lambda(R)A]\}, \quad (2)$$

where the summation runs over irreducible representations of the gauge group, U_1 and U_2 are the path-ordered

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exponentials of the gauge field on the boundaries, χ is the character of the group element, and A is the area of the surface. C_Λ is some function related to Λ. Taking C_Λ to be a linear function of the rescaled Casimirs of the gauge group U(N),

$$\tilde{C}_l(R) := \frac{1}{N^{l+1}} \sum_{i=1}^{N} (n_i + N - i)^l, \quad (3)$$

where n_i's are non-increasing functions of i characterizing the representation (the Young tableau), one defines a function W by

$$-N^2 W \left[A \sum_l a_l \tilde{C}_l(R) \right] := w[A C_\Lambda(R)]. \quad (4)$$

In the large-N limit, the exponential in (2) becomes

$$\exp\{w[C_\Lambda(R)A]\} = \exp\left\{ -N^2 W \left[A \int_0^1 \mathrm{d}x G(\phi) \right] \right\}, \quad (5)$$

where

$$G(\phi) := \sum_l (-1)^l a_l \phi^l. \quad (6)$$

Also, following [5],

$$\phi := \frac{i - n_i - N}{N}, \quad (7)$$

and

$$x := \frac{i}{N}. \quad (8)$$

Following [11], one can write the characters in (2) as a function of $\phi(x)$, and the eigenvalue densities $\sigma_1(\theta)$ and $\sigma_2(\theta)$ of the boundary matrices U_1 and U_2. Then, for large N, (2) is written as

$$Z = \int \mathrm{D}\phi \exp\left\{ -N^2 W \left[A \int_0^1 \mathrm{d}x G(\phi) \right] + N^2 \Gamma[\phi, \sigma_1, \sigma_2] \right\}. \quad (9)$$

Note that the exponent in (9) consists of two parts. The first part depends on both W and G. The second part, coming from the characters, depends on neither W nor G. For $N \to \infty$, the wave function (9) is determined by the representation maximizing the exponent. This representation satisfies

$$-AW' \left[A \int_0^1 \mathrm{d}x G(\phi) \right] G'[\phi(x)] + \frac{\delta \Gamma}{\delta \phi(x)} = 0. \quad (10)$$

Defining

$$\tilde{A} := AW' \left[A \int_0^1 \mathrm{d}x G(\phi) \right], \quad (11)$$

it is obvious that this equation is equivalent to the equation determining the dominant representation in

$$\tilde{Z} = \int \mathrm{D}\phi \exp\left\{ -N^2 \tilde{A} \int_0^1 \mathrm{d}x G(\phi) + N^2 \Gamma[\phi, \sigma_1, \sigma_2] \right\}. \quad (12)$$

But the dominant representation of this has been found in [12]. Defining the Young tableau density [5]

$$\rho(\phi) := \frac{\mathrm{d}x}{\mathrm{d}\phi}, \quad (13)$$

it has been shown in [12] that in order to obtain the Young tableau density corresponding to the dominant representation, one should solve the generalized Hopf equation

$$\frac{\partial}{\partial t}(v \pm \mathrm{i}\pi\sigma) + \frac{\partial}{\partial \theta} G[-\mathrm{i}(v \pm \mathrm{i}\pi\sigma)] = 0, \quad (14)$$

with the boundary conditions

$$\sigma(t = 0, \theta) = \sigma_1(\theta),$$
$$\sigma(t = \tilde{A}, \theta) = \sigma_2(\theta). \quad (15)$$

Then, if there exists some t_0 for which

$$v(t_0, \sigma) = 0, \quad (16)$$

one denotes the value of σ for $t = t_0$ by σ_0:

$$\sigma_0(\theta) := \sigma(t_0, \theta), \quad (17)$$

and the desired density satisfies

$$\pi\rho[-\pi\sigma_0(\theta)] = \theta. \quad (18)$$

What is shown is that from this point of view, the non-local theory behaves like a local theory but with a surface area $\tilde{A}$ instead of A. Note, however, that $\tilde{A}$ itself depends on the Young tableau density of the dominant representation, through (11) or equivalently

$$\tilde{A} = AW' \left[A \int \mathrm{d}z \rho(z) G(z) \right]. \quad (19)$$

A special case of this result was obtained in [10], where non-local generalized Yang–Mills theories on the sphere were studied. It was shown there that in the limit $N \to \infty$, the theory is like a local generalized Yang–Mills theory with the surface area $\tilde{A}$ instead of A. The dependence of $\tilde{A}$ on A and ρ was the same as (19).

3 The critical behavior of the non-local Yang–Mills theory

A non-local Yang–Mills theory is defined by

$$G(\phi) = \frac{1}{2}\phi^2. \quad (20)$$

In [11], the critical area for a Yang–Mills theory on a disk, $\sigma_1(\theta)=\delta(\theta)$, has been found to be

$$A_{\rm cr}^{-1}=\frac{1}{\pi}\int\frac{{\rm d}\theta'\sigma_2(\theta')}{\pi-\theta'}. \tag{21}$$

For a sphere, $\sigma_2(\theta)=\delta(\theta)$, and one arrives at the familiar result

$$A_{\rm cr}=\pi^2. \tag{22}$$

These results can be used to obtain the critical area for a non-local Yang–Mills theory on a disk. One can obtain $\tilde{A}_{\rm cr}$:

$$\tilde{A}_{\rm cr}^{-1}=\frac{1}{\pi}\int\frac{{\rm d}\theta'\sigma_2(\theta')}{\pi-\theta'}. \tag{23}$$

There remains, however, one problem. To obtain $A_{\rm cr}$ from $\tilde{A}_{\rm cr}$, using (19), one needs the critical density $\rho_{\rm cr}$. Even for the disk, it is not easy to find a closed form for $\rho_{\rm cr}$ for arbitrary σ_2. On the sphere, the situation is better. In [11] it has been shown that the solution to the Hopf equation for $\sigma_1(\theta)=\sigma_2(\theta)=\delta(\theta)$ is

$$\pi\sigma(t,\theta)=\frac{\tilde{A}}{2t(\tilde{A}-t)}\sqrt{\frac{4t(\tilde{A}-t)}{\tilde{A}}-\theta^2}, \tag{24}$$

and

$$v(t,\theta)=\frac{(2t-\tilde{A})\theta}{2t(t-\tilde{A})}. \tag{25}$$

From this, one finds

$$t_0=\frac{\tilde{A}}{2}. \tag{26}$$

Inserting this in (24), one arrives at

$$\pi\sigma_0(\theta)=\frac{2}{\tilde{A}}\sqrt{\tilde{A}-\theta^2}. \tag{27}$$

So, using (18),

$$\rho(z)=\frac{\tilde{A}}{2\pi}\sqrt{\frac{4}{\tilde{A}}-z^2}. \tag{28}$$

At the critical area, the maximum of ρ becomes 1. This shows that

$$\tilde{A}_{\rm cr}=\pi^2, \tag{29}$$

as expected. But now one can insert the critical density in (19) to obtain

$$\pi^2=A_{\rm cr}W'\left(\frac{A_{\rm cr}}{\pi^2}\right). \tag{30}$$

This is in accordance with what was found in [10].

One can go further. Consider a disk with the boundary condition

$$\sigma_2(\theta)=\frac{2}{\pi s^2}\sqrt{s^2-\theta^2}. \tag{31}$$

The solution to the Hopf equation with this boundary condition is easily obtained using the solution to the Hopf equation for the boundary conditions corresponding to the sphere. One finds

$$\pi\sigma(t,\theta)=\frac{A_0}{2t(A_0-t)}\sqrt{\frac{4t(A_0-t)}{A_0}-\theta^2} \tag{32}$$

and

$$v(t,\theta)=\frac{(2t-A_0)\theta}{2t(t-A_0)}, \tag{33}$$

where A_0 is defined through

$$\frac{4\tilde{A}(A_0-\tilde{A})}{A_0}:=s^2, \tag{34}$$

or

$$A_0:=\frac{4\tilde{A}^2}{4\tilde{A}-s^2}. \tag{35}$$

Again, one sets $v=0$ to obtain σ_0:

$$\pi\sigma_0(\theta)=\frac{2}{A_0}\sqrt{A_0^2-\theta^2}. \tag{36}$$

From this,

$$\rho(z)=\frac{A_0}{2\pi}\sqrt{\frac{4}{A_0}-z^2}. \tag{37}$$

Note that for the specific boundary condition (31), the shape of the Young tableau density ρ is always the semi-ellipse function obtained for the sphere. At the critical point,

$$\rho_{\rm cr}(z)=\frac{\pi}{2}\sqrt{\frac{4}{\pi^2}-z^2}. \tag{38}$$

Again, this is universal, as long as the boundary condition is like (31). Putting this in (19), one obtains

$$\tilde{A}_{\rm cr}=A_{\rm cr}W'\left(\frac{A_{\rm cr}}{\pi^2}\right). \tag{39}$$

To find $A_{\rm cr}$, one needs $\tilde{A}_{\rm cr}$, which is obtained from (34), and using $A_{0,{\rm cr}}=\pi^2$:

$$\tilde{A}_{\rm cr}=\left(\frac{\pi^2}{2}\right)(1+\sqrt{1-s^2/\pi^2}). \tag{40}$$

Combining this with (39), one arrives at

$$\frac{1+\sqrt{1-s^2/\pi^2}}{2}=\frac{A_{\rm cr}}{\pi^2}W'\left(\frac{A_{\rm cr}}{\pi^2}\right). \tag{41}$$

What is achieved till now, is that we obtained the critical density for the sphere and for a disk with certain boundary conditions making the disk *a part of a sphere*. The critical area can also be found for a cylinder which is *a part of a sphere*. Consider the boundary conditions

$$\sigma_1(\theta)=\frac{2}{\pi s_1^2}\sqrt{s_1^2-\theta^2},$$
$$\sigma_2(\theta)=\frac{2}{\pi s_2^2}\sqrt{s_2^2-\theta^2}. \tag{42}$$

One can use (32) and (33) as the solutions to the Hopf equation, but with (34) replaced by

$$\frac{4t_1(A_0 - t_1)}{A_0} := s_1^2, \quad \frac{4t_2(A_0 - t_2)}{A_0} := s_2^2, \tag{43}$$

and

$$\tilde{A} = t_2 - t_1. \tag{44}$$

Following the same arguments as used for the disk, one obtains

$$t_1 = \left(\frac{\pi^2}{2}\right)\left(1 - \sqrt{1 - s_1^2/\pi^2}\right), \quad t_2 = \left(\frac{\pi^2}{2}\right)\left(1 + \sqrt{1 - s_2^2/\pi^2}\right), \tag{45}$$

and

$$\tilde{A}_{\rm cr} = \left(\frac{\pi^2}{2}\right)\left(\sqrt{1 - s_2^2/\pi^2} + \sqrt{1 - s_1^2/\pi^2}\right). \tag{46}$$

Using this, the critical area is found to satisfy

$$\frac{\sqrt{1 - s_2^2/\pi^2} + \sqrt{1 - s_1^2/\pi^2}}{2} = \frac{A_{\rm cr}}{\pi^2} W'\left(\frac{A_{\rm cr}}{\pi^2}\right). \tag{47}$$

The last thing to be considered is the case of a disk which is *almost a sphere*; that is, a disk with the boundary condition $\sigma_2(\theta) \approx \delta(\theta)$. By this approximation it is meant that σ_2 is an even function and one takes into account only the second moment of θ:

$$r := \int \mathrm{d}\theta \sigma_2(\theta)\theta^2. \tag{48}$$

It is assumed that σ_2 is narrowly localized around $\theta = 0$, so that one can neglect the effect of the higher moments of θ. As only the second moment of θ is important, one can approximate σ_2 with (31) for a small value of s. This value of s is related to r through

$$r = \int_{-s}^{s} \mathrm{d}\theta\theta^2 \left(\frac{2}{\pi s^2}\sqrt{s^2 - \theta^2}\right) = \frac{s^2}{4}. \tag{49}$$

One can substitute this value of s in (41) to obtain

$$1 - \frac{r}{\pi^2} = \frac{A_{\rm cr}}{\pi^2} W'\left(\frac{A_{\rm cr}}{\pi^2}\right). \tag{50}$$

An exactly similar argument can be used for a cylinder with the boundary conditions near a delta function. The result would be

$$1 - \frac{r_1 + r_2}{\pi^2} = \frac{A_{\rm cr}}{\pi^2} W'\left(\frac{A_{\rm cr}}{\pi^2}\right), \tag{51}$$

where

$$r_i := \int \mathrm{d}\theta \sigma_i(\theta)\theta^2. \tag{52}$$

Similar arguments may work for special boundary conditions and non-local generalized Yang–Mills theories, provided the dominant representation of the system is known for a sphere.

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